ON THE STABLE SET OF ASSOCIATED PRIME IDEALS OF A MONOMIAL IDEAL

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Abstract. It is shown that any set of nonzero monomial prime ideals can be realized as the stable set of associated prime ideals of a monomial ideal. Moreover, an algorithm is given to compute the stable set of associated prime ideals of a monomial ideal.

Introduction

It is known by a result of Brodmann [1] that in any Noetherian ring the set of associated prime ideals $\mathrm{Ass}(I^s)$ for the powers of an ideal I stabilizes for $s \gg 0$. In other words, there exists an integer s_0 such that $\mathrm{Ass}(I^s) = \mathrm{Ass}(I^{s+1})$ for all $s \geq s_0$. This stable set of associated prime ideals is denoted by $\mathrm{Ass}^\infty(I)$. In recent years there have been several publications [3, 4, 8, 9], mostly in combinatorial contexts, to describe $\mathrm{Ass}^\infty(I)$.

The main purpose of this note is to show that for any set of nonzero monomial prime ideals, there exists a monomial ideal for which this given set is precisely the set of stable associated prime ideals. This result is given in Theorem 1.2. Describing the possible stable sets of associated prime ideals for squarefree monomial ideals remains an open problem.

In Section 2 we give an algorithm that determines $\mathrm{Ass}^\infty(I)$ for any monomial ideal. The routine written in *Macaulay 2* can be found in [2].

1. The set $\mathrm{Ass}^\infty(I)$ of a monomial ideal

For the proof of the main result of this section we first need the following general fact about associated prime ideals.

Lemma 1.1. *Let R be a Noetherian ring, $I \subset R$ an ideal and $P \subset R$ a prime ideal such that $P \not\subseteq P'$ for all $P' \in \mathrm{Ass}(I)$. Let k be an integer such that $I \not\subseteq P^{(k)}$, where $P^{(k)}$ denotes the kth symbolic power of P. Then $\mathrm{Ass}(I \cap P^{(k)}) = \mathrm{Ass}(I) \cup \{P\}$.*

Proof. Let $I = Q_1 \cap Q_2 \cap \ldots \cap Q_m$ be a irredundant primary decomposition of I with $\sqrt{Q_i} = P_i$ and $P_i \neq P_j$ for $i \neq j$. We show that $I \cap P^{(k)} = Q_1 \cap Q_2 \cap \ldots \cap Q_m \cap P^{(k)}$ is a irredundant primary decomposition of $I \cap P^{(k)}$. Considering the fact $I \not\subseteq P^{(k)}$, it is enough to show that

$$(1) \qquad (\bigcap_{j\neq i} Q_j) \cap P^{(k)} \not\subseteq Q_i$$

for all i. Since $I = Q_1 \cap Q_2 \cap \ldots \cap Q_m$ is a irredundant primary decomposition of I, there exists an element $a \in (\bigcap_{j\neq i} Q_j) \setminus Q_i$. There also exists an element $b \in P \setminus P_i$ by our

1991 *Mathematics Subject Classification.* 13C13, 13P99.

Key words and phrases. Monomial ideals, associated prime ideals, powers of ideals.

This paper was written while S. Bayati and G. Rinaldo were visiting Universität Duisburg-Essen, Campus Essen. The first author was supported by the Ministery of Science of Iran, the third author by INDAM GNSAGA.

assumption. So $ab^k \in (\bigcap_{j\neq i} Q_j) \cap P^{(k)}$, but $ab^k \notin Q_i$. Indeed, if $ab^k \in Q_i$ it follows that $a \in Q_i$ or $b^k \in \sqrt{Q_i} = P_i$, a contradiction. This implies (1). $\square$

Now we consider a more specific situation. Let K be a field, $S = K[x_1, \ldots, x_n]$ the polynomial ring in the variables $x_1, \ldots, x_n$ and $I \subset S$ a monomial ideal. As usual we denote by $G(I)$ the unique minimal set of monomial generators of I.

The associated prime ideals of I are all monomial prime ideals, that is, ideals of the form $P_F = (\{x_i\colon\ i \in F\})$ where $F \subset [n]$. The main result of this section is the following

Theorem 1.2. *Let $P_1, \ldots, P_m \subset S$ be an arbitrary collection of nonzero monomial prime ideals. Then there exists a monomial ideal $I \subset S$ such that*

$$\mathrm{Ass}^\infty(I) = \{P_1, \ldots, P_m\}.$$

Proof. We may assume that $P_i \neq P_j$ for $i \neq j$, and that $|G(P_i)| \leq |G(P_j)|$ for $i < j$. For $r = 1, \ldots, m$, we inductively define the monomial ideals J_r with $\mathrm{Ass}(J_r) = \{P_1, \ldots . P_r\}$ as follows: $J_1 = P_1$. Suppose that for some $r < m$ the ideal J_r is already defined. Then we set

$$J_{r+1} = J_r \cap P_{r+1}^{k_{r+1}},$$

where $k_{r+1} > \min\{\deg u\colon\ u \in G(J_r)\}$. By Lemma 1.1 we have that

$$\mathrm{Ass}(J_{r+1}) = \{P_1, \ldots, P_{r+1}\}.$$

Indeed, since $|G(P_i)| \leq |G(P_j)|$ for $i < j$, it follows that $P_{r+1} \not\subseteq P_i$ for $i \leq r$. Moreover, $P_{r+1}^{(k_{r+1})} = P_{r+1}^{k_{r+1}} \not\supseteq J_r$, by the choice of k_{r+1}.

Let t be any positive integer. Then tk_{r+1} is bigger than the minimum degree of $P_1^{tk_1} \cap P_2^{tk_2} \cap \ldots \cap P_r^{tk_r}$ for $r = 1, \ldots, m-1$, because $J_r^t \subset P_1^{tk_1} \cap P_2^{tk_2} \cap \ldots \cap P_r^{tk_r}$. Thus as before we see that

$$\mathrm{Ass}(P_1^{tk_1} \cap P_2^{tk_2} \cap \ldots \cap P_m^{tk_m}) = \{P_1, \ldots, P_m\} \quad \text{for all} \quad t. \tag{2}$$

By [7, Corollary 2.2], there exists an integer d such that for

$$I = P_1^{dk_1} \cap P_2^{dk_2} \cap \ldots \cap P_m^{dk_m}$$

we have

$$I^s = P_1^{sdk_1} \cap P_2^{dk_2} \cap \ldots \cap P_m^{sdk_m} \quad \text{for all} \quad s \geq 1.$$

Thus (2) implies that $\mathrm{Ass}(I) = \mathrm{Ass}(I^s) = \{P_1, \ldots, P_m\}$. This yields the desired conclusion. $\square$

The monomial ideal I that we constructed in Theorem 1.2 has the property that $\mathrm{Ass}(I) = \mathrm{Ass}^\infty(I)$. In general, for any ideal I one has $\mathrm{Min}(I) = \mathrm{Min}(I^s)$ for all s. Hence as strengthening of Theorem 1.2, one could ask the following question: suppose we are given two sets $A = \{P_1, \ldots, P_\ell\}$ and $B = \{P'_1, \ldots, P'_m\}$ of monomial prime ideals such that the minimal elements of these sets with respect to inclusion are the same. For which such sets does exist a monomial ideal I such that $\mathrm{Ass}(I) = A$ and $\mathrm{Ass}^\infty(I) = B$? For example, there is no monomial ideal I with $\mathrm{Ass}(I) = \{(x_1), (x_2)\}$ and $\mathrm{Ass}^\infty(I) = \{(x_1), (x_2), (x_1, x_2)\}$.

It would be also interesting to understand the possible sets of prime ideals for $\mathrm{Ass}^\infty(I)$ when I is a *squarefree* monomial ideal. Consider for example the set of monomial prime ideals $\{(x_1), (x_2), (x_1, x_2)\}$. This set cannot be $\mathrm{Ass}^\infty(I)$ for any squarefree monomial ideal I. Thus there is no analogue of Theorem 1.2 for squarefree monomial ideals.

2. An algorithm to compute $\mathrm{Ass}^\infty(I)$

In this section we describe an algorithm that we implemented in *Macaulay 2* (see [2]) to compute $\mathrm{Ass}^\infty(I)$ for a monomial ideal I. Since there are only finitely many monomial prime ideals in S, it is enough to have an algorithm to determine whether a monomial prime ideal $P \subset S$ belongs to $\mathrm{Ass}^\infty(I)$.

The input of our algorithm is a monomial ideal I given by its set of monomial generators $G(I)$, and a monomial prime ideal $P = (x_{i_1}, \ldots, x_{i_k})$.

Step 1: Define the ideal $J \subset S' = K[x_{i_1}, \ldots, x_{i_k}]$ generated by the monomials u^* with $u \in G(I)$, where u^* is obtained from u by the substitution $x_j \mapsto 1$ for $x_j \notin P$.

Step 2: If $J = S'$, then $P \notin \mathrm{Ass}^\infty(I)$ (actually $I \not\subseteq P$). Else go to Step 3.

Step 3: Form the Rees algebra $\mathcal{R}(J)$ and compute the Koszul homology

$$H = H_{k-1}(x_{i_1}, \ldots, x_{i_k}; \mathcal{R}(J)).$$

Step 4: Compute the Krull dimension of H. If $\dim H > 0$, then $P \in \mathrm{Ass}^\infty(I)$; else $P \notin \mathrm{Ass}^\infty(I)$.

Indeed, to justify this algorithm notice that $P \in \mathrm{Ass}^\infty(I)$ if and only if $PS_x \in \mathrm{Ass}(J^s S_x)$ for all $s \gg 0$, where $x = \prod_{x_i \notin P} x_i$. This is equivalent to say that $\operatorname{depth} S'/J^s = 0$ for all $s \gg 0$. Next we observe that the j-th Koszul homology $H_j(x_{i_1}, \ldots, x_{i_k}; (J))$ is a graded $\mathcal{R}(J)$-algebra with

$$H_j(x_{i_1}, \ldots, x_{i_k}; \mathcal{R}(J))_s = H_j(x_{i_1}, \ldots, x_{i_k}; J^s) \quad \text{for all} \quad s.$$

Hence we have $\operatorname{depth} S'/J^s = 0$ if and only if $H_{k-1}(x_{i_1}, \ldots, x_{i_k}; \mathcal{R}(J))_s \neq 0$. Therefore $\operatorname{depth} S'/J^s = 0$ for all $s \gg 0$ if and only if the finitely generated $\mathcal{R}(J)$-module $H = H_{k-1}(x_{i_1}, \ldots, x_{i_k}; \mathcal{R}(J))$ has infinitely many nonzero components, and this is the case if and only the Krull dimension of H is positive.

To demonstrate this algorithm we consider the following computation in *Macaulay 2*. We want to check whether $P = (a, b, c, e, f) \in \mathrm{Ass}^\infty(I)$ for $I = (a^3b^2e, bc^3d, b^4de^2f, ab^2cf^3)$.

```
S=QQ[a..f];
J=monomialIdeal(a^3*b^2*e,b*c^3,b^4*e^2*f, a*b^2*c*f^3);
R=reesAlgebra(J);
P={a,b,c,e,f};
phi=matrix{P};
C=(koszul phi)**R;
dim(HH_4 C)>0
```

In this case the output is: `true`. This means that $P \in \mathrm{Ass}^\infty(I)$. Other examples and the source file for a routine can be found in [2] where $\mathrm{Ass}^\infty(I)$ is computed for monomial ideals.

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